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Portfolio Optimization: Indifference Curve Approach

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ABSTRACT

The study examines the monthly stock prices of 45 SENSEX companies for the period ranging from February 2002 to January 2012. Also the study includes the Indian G-SEC long term bonds with maturities ranging from 15 to 25 years. The set of all efficient portfolios is called the efficient frontier. All risk-averse investors who act to maximize expected utility have an optimal portfolio on this frontier. Based on the risk-aversion factor and the investment time horizon of each individual investor, an attempt is being made to select the optimal portfolio for that particular investor. Given a utility function for an individual investor, the portfolio optimization problem is to find the indifference curve which is tangent to the efficient frontier. The optimal portfolio for the investor lies at the point of tangency between the efficient frontier and the indifference curve. The findings of the study bring out the importance of the investor"s time horizon and the risk-aversion factor in portfolio optimisation.

Keywords: *Efficient Frontier; Indifference Curve; Risk-aversion Factor;Investment Time Horizon; Portfolio Optimization.*

1.0 Introduction

Portfolio theory was first discovered and developed by Harry Markowitz in the 1950's (Markowitz, 1952). Markowitz formulated the portfolio problem as a choice of the mean and variance of a portfolio of assets. His work forms the foundation of modern finance. Post Markowitz theory related to portfolio is often called „Modern Portfolio Theory".

Markowitz approach is based on the mean-variance analysis, where the variance of the overall rates of return is taken as a risk measure and expected value measures profitability. In contrast to expected utility maximisation, mean – variance analysis takes into account only the first two moments and there is no clear theoretical foundation. The special assumption under which mean- variance analysis is consistent with expected utility maximisation is the assumption of Von Neumann- Morgensten utility. [Modern portfolio Theory: Some main Results, Heinz H. Muller, Astin Bulletin, Vol18, no.2] Heinz H. Muller, in his article has summarised some main results in modern portfolio. He discusses Markowitz approach which shows that due to the correlation between the returns of the financial assets, diversification allows in general, only for a reduction but not for the elimination of the risk. Markowitz (1952) was the first who took the covariance between the rates of return into account. John Norstad (1999, updated 2001) in his paper, „An introduction to portfolio theory" has introduced the basic concept of portfolio theory, including the notions of efficiency, risk-return graphs, the efficient frontier iso-utility curve and asset allocation optimisation problems. He has developed the theory in both, a simplified setting by assuming returns are normally distributed over the time period and considering Random walk Model where returns are log-normally distributed. The assumption that the returns are normally distributed along with the assumption of negative exponential utility leads to portfolio maximisation problem. The log normal

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distribution in Random walk model is better approximation to the distribution of observed historical returns for common financial assets like stocks and bonds. Log normal returns are also consistent with Central Limit Theorem. In Random walk model, portfolio efficiency is determined by instantaneous expected returns and the standard deviations of these results. The additional assumption of Iso elastic-utility leads to portfolio optimisation problem that are linear in return and variance.

Researchers (Elton, Gruber, 1997) believe that there are two reasons for the persistence of mean variance theory. First, mean variance theory itself places large data requirements on the investor, and there is no evidence that adding additional moments improves the desirability of the portfolio selected. Second, the implications of mean variance portfolio theory are well developed, widely known, and have great intuitive appeal.

Mean variance portfolio theory was developed to find the optimum portfolio when an investor is concerned with return distributions over a single period. MPT models an asset's return as a normally distributed function, defines risk as the standard deviation of return, and models a portfolio as a weighted combination of assets, so that the return of a portfolio is the weighted combination of the assets' returns. The primary principle upon which Modern Portfolio Theory (MPT) is based is the Random Walk Hypothesis which states that the movement of asset prices follows an unpredictable path. MPT assumes that investors are risk averse and rational. The implication is that a rational investor will not invest in a portfolio if a second portfolio exists with a more favourable risk-expected return profile – i.e., if for that level of risk an alternative portfolio exists which has better expected returns.

In Markowitz approach every mean-variance efficient portfolio is a combination of the riskless investment with reference to portfolio consisting of risky assets. It is important to note that special structure of the set of mean- variance, efficient portfolios provide the basis of the Capital Asset Pricing Model (Heinz H., Muller, et.al)

The Capital Asset Pricing Model (CAPM) is used to determine the expected rate of return of an asset. Sharpe - Lintner CAPM model assumes that there is a risk-less asset. The model takes into account the asset‟s sensitivity to non-diversifiable risk, often represented by Beta (β) as well as the expected return of the market and expected return of theoretical risk free asset. The assumption of existence of riskless asset is somewhat questionable, especially if one is interested in real return. The Black (1972) model derives the CAPM relationship without assuming the existence of riskless asset. The CAPM relationship obtained from Sharpe-Lintner model can be written as

$$E(R_i) - R_f = \frac{Cov\ R_i R_m}{Var\ R_m} [E(R_m - R_f)]$$

$$E(R_i) = R_f + \beta_i * E[R_m - R_f] \quad (1)$$

Where β_i is called the beta coefficient of assets

$E(R_i) - R_f$ is the risk premium on asset i.

$E(R_m - R_f)$ is risk premium on the market portfolio.

Indifference Curves defines the utility of the investors. For any given curve, the possible investments which plot on the curve will have the same expected utility, and the investor is therefore indifferent among them. The y-intercept of a curve is the value of **k**. This is the investor's certainty equivalent for all the other possible investments on the curve. The higher indifference curves have larger certainty equivalents and larger expected utility. Assuming that the investor has a negative exponential utility function, measured as a function of return rather than end-of-period wealth, the investor maximizes the following function over the feasible set or, over the efficient frontier:

$$k = R_p - ½[(A\sigma^2_{pn}) / n] \quad (2)$$

Where, k gives the utility of the portfolio which is constant on a particular indifference curve. „A‟ denotes the coefficient of risk-aversion, and σ^2_{pn} stands for the variance of portfolio for n-year.

The objective of this paper is to select an optimal portfolio, from a given set of portfolios, based on the investor‟s risk-aversion factor and the investment time horizon. Risk-aversion factor denotes the amount of risk that a particular investor is willing to tolerate. Investment time horizon defines the time-period for which a particular investor wishes to invest in the portfolio. The study is based on a simplified assumption that returns are normally distributed over the time period.

2.0 Research Methodology

In this paper we have considered the feasible set of investment alternatives consisting of all portfolios combining long-term G-SEC bonds and 45 SENSEX stocks. It is assumed that leverage and short sales are not allowed, so each portfolio consists of some percentage x of bonds and 100-x of stocks where,

$0 < x < 100$

The data set consists of monthly adjusted closing prices of the market (SENSEX), stock (45 stocks of SENSEX), and risk-free rate of return (from MIBOR) for the period from February 2002 through January 2012 (viz. 10 years). The sample period exhibits a mixed set of economic environment in Indian economy. Long-term GSEC bonds are the Government Securities with years of maturity ranging from 15 years to 20 years. Such bonds are considered to be the risk-free assets. In addition, the choice of the optimal portfolio is based on the investors" risk-aversion factor and the investment time horizon. The investment time horizon varies from one year to five years. It is assumed that two types of investors exist inthe market. The first category includes the investors with low risk-aversion factor, i.e. investors" who can bear more risk for higher expected returns. The second category of investors includes the one with high risk- aversion factor, i.e. investors" who are not willing to take large amounts of risk. So, the risk-aversion factor is chosen to be A=4 for investors who can bear more risk and A=40 for the investors who cannot take large amounts of risk.

In the present study, the capital asset pricing model (CAPM) is used to determine the expected rate of return on a stock. The model takes into account the asset's sensitivity to non-diversifiable risk, often represented by the quantity beta (β), as well as the expected return of the market and the expected return of a theoretical risk-free asset. The CAPM equation for the Security Market Line is given as

$$E(R_i) = R_f + \beta_i * [R_m - R_f] \quad (3)$$

Where, $E(R_i)$ is the expected return on security,

R_f gives the risk-free rate, β_i denotes the systematic risk, and R_m is the expected return on market portfolio. Market Returns are the returns that the investors generate out of the stock market. Here it has been calculated as below:

$$R_m = [(P_t - P_{t-1})/P_{t-1}]*100 \quad (4)$$

Where, P_t denotes the price of index in time period t, and P_{t-1} denotes the price of index in preceding time period t-1.A stock's market price is a function of the market's perception of the value of the future profits a company can create. Symbolically, return on stocks can be written as

$$R_s = ((P_t - P_{t-1})/P_{t-1})*100 \quad (5)$$

Where, R_s is the Percentage return on a stock.

Beta is a measure of the volatility, or systematic risk, of a security or a portfolio in comparison to the market. Beta is calculated using regression analysis.

Symbolically

$$\beta = [Cov(R_s, R_m)]/\sigma_m^2] \quad (6)$$

Where, Cov (R_s, R_m) gives the covariance between security and market return and σ_m^2 gives the variance of market return. Return on bonds is calculated using Yield to Maturity (YTM) in Microsoft excel.

YTM = Rate (NPER, PMT, PV, FV) (7)

Where, NPER denotes total number of payments, PMT denotes payment made each period, PV denotes present value, and FV denotes future value.

Risk for bonds is calculated in the same manner as the calculation of risk of stocks. Symbolically

$$\sigma = \sqrt{\frac{\sum(\bar{x} - x)^2}{n - 1}}$$

.Where, σ gives the standard deviation (risk), X is the expected return, and n is used to denote number of observations.

Further, 21 feasible portfolios are constructed using the mix of stocks & bonds. The most conservative portfolio consists of 0% stocks & 100% bonds whereas the most aggressive portfolio consists of 100% stocks & 0% bonds. In between the conservative and aggressive portfolio the portfolios are assigned weights like 5% stocks & 95% bonds, 10% stocks & 90% bonds and so on

Portfolio return is the proportion-weighted combination of the constituent assets' returns. Symbolically

$$R_P = W_S R_S + W_B R_B \quad (8)$$

Where, R_P is the portfolio return, W_S and W_B gives the weight assigned to stocks and bonds, and R_S and R_B gives the return on stocks and bonds.

The feasible set of stocks and bonds plots as a curve between the risk and the expected return. The most conservative portfolio, 100% bonds, has both lower expected return and lower risk than does the most aggressive portfolio, 100% stocks.

All of the portfolios which are more aggressive than the minimum variance portfolio are efficient. The set of all efficient portfolios is called the efficient frontier. All risk-averse investors who act to maximize expected utility have an optimal portfolio on this frontier.

The slope of the Efficient Frontier at any point depicts how much extra expected return is obtained bytaking some more risk.

This is called the Return/Risk Trade-off. It is a Measure of the Risk Adjusted Return and is also called as the Sharpe Ratio (S).

Return/Risk Trade-off = change in RP / change in σ_P

2.1 Result Analysis

Consider two investors; one with Risk Aversion A=4 and another with Risk Aversion A=40. The investor with A=4 is more risk tolerant (less risk averse) than the investor with A=40.

3.0 Based on Investment Time Horizon

Assuming the risk aversion factor, A=4k

$$k = R_P - ½[(4*\sigma^2_{Rn}) / n] \quad (9)$$

Case I: - Investment Time Horizon, n = 1-year

$$R_P = k + (2*\sigma^2_{R1}) \quad (10)$$

Since the optimal portfolio lies at the point of tangency of the efficient frontier and the indifference curve, consider the curve with k=5.90%.

This curve is tangent to the feasible set curve at approximately the portfolio which is 80% bonds and 20% stocks. This portfolio maximizes the investor's expected utility.

Fig: 1. Optimal Portfolio (n=1, A=4)

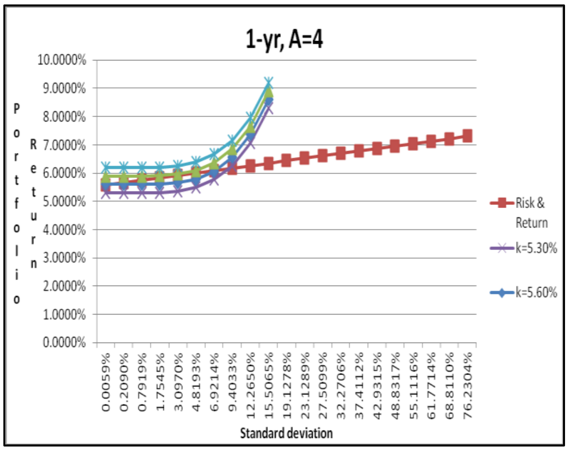
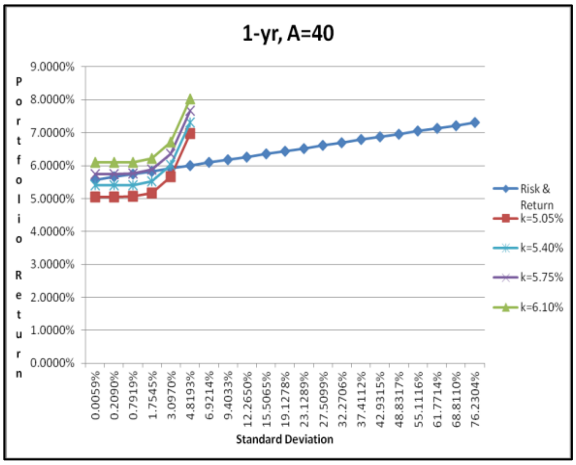


Fig: 2. Optimal Portfolio (n=1, A=40)



Case II: - Investment Time Horizon, n = 2-year

$$R_P = k + \sigma^2_{R2} \quad (11)$$

Consider the curve with k=5.90%. The optimal portfolio for our investor is clearly the 75% bond/25%stock portfolio, with a certainty equivalent of 5.90%. It is thus clear that as the investment time horizon increases, the indifference curve becomes a little flatter. Also, the investor is willing to accept a larger quantity of risk.

Case III: - Investment Time Horizon, n = 3-year

$$R_P = k + [(2*\sigma^2_{R3}) / 3] \quad (12)$$

The curve with k=6.00% is tangent to the feasibleset curve at approximately the portfolio which is 70% bonds and 30% stocks. This portfolio is preferred to any of the portfolios on the bottom and second-to- bottom indifference curves.

Fig: 3. Optimal Portfolio (n=2, A=4)

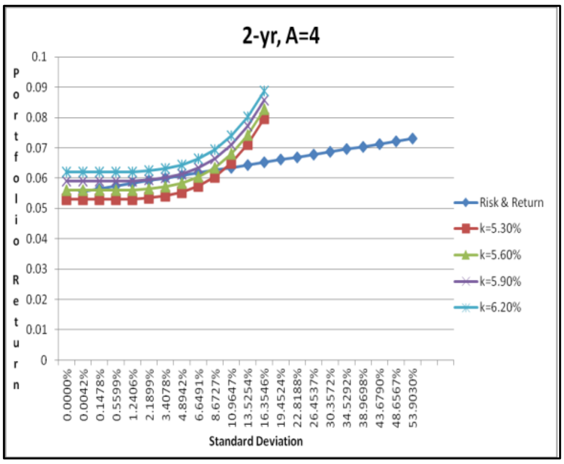
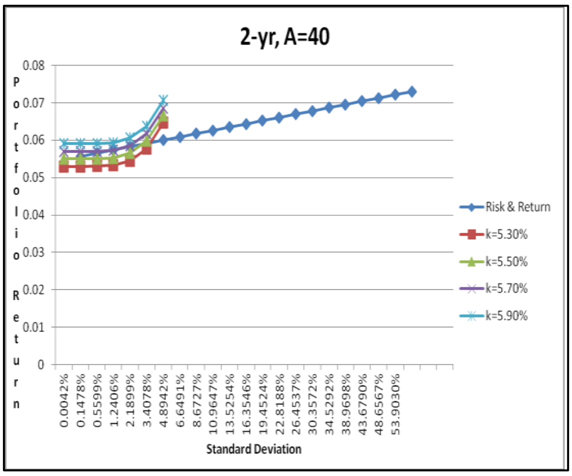


Fig: 4. Optimal Portfolio (n=2, A=40)



Case IV: - Investment Time Horizon, n = 4-year

$$R_P = k + (\sigma^2_{R4} / 2) \quad (13)$$

At k=6.10%, the point of tangency lies at approximately the portfolio which is 60% bonds and 40% stocks. This portfolio maximizes the investor's expected utility

Case V: - Investment Time Horizon, n = 5-year

$$R_P = k + [(2*\sigma^2_{R5}) / 5] \quad (14)$$

Considering the curve with k=6.20%, the optimal portfolio for our investor is clearly the 55% bond/45%stock portfolio

Fig: 5. Optimal Portfolio (n=3, A=4)

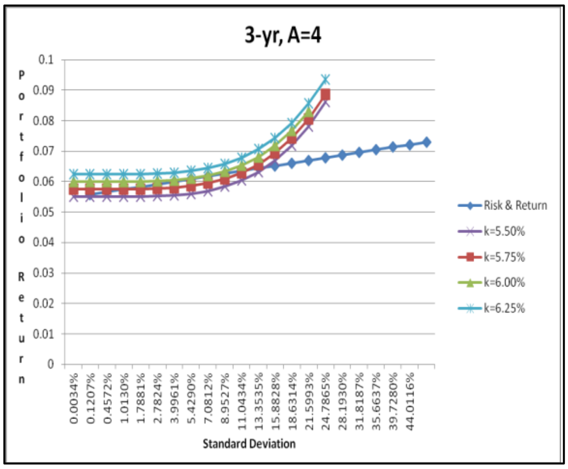


Fig: 6. Optimal Portfolio (n=3, A=40)

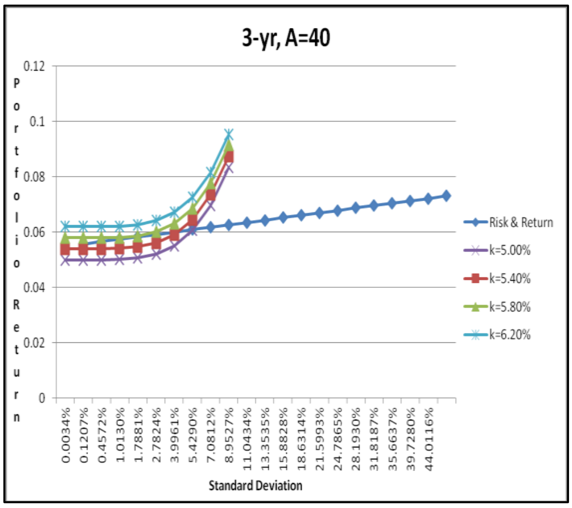


Fig: 7. Optimal Portfolio (n=4, A=4)

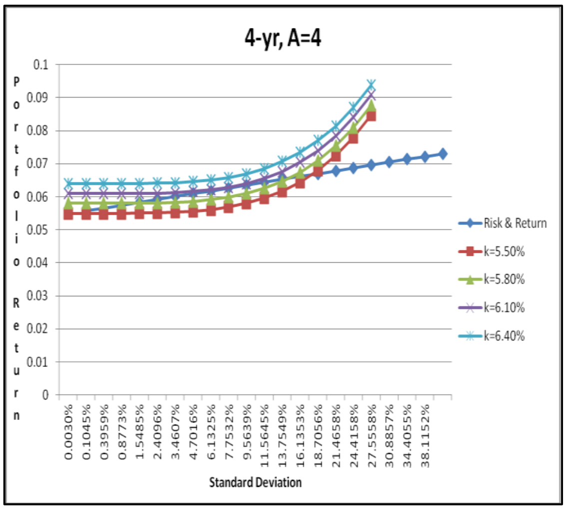


Fig: 8. Optimal Portfolio (n=4, A=40)

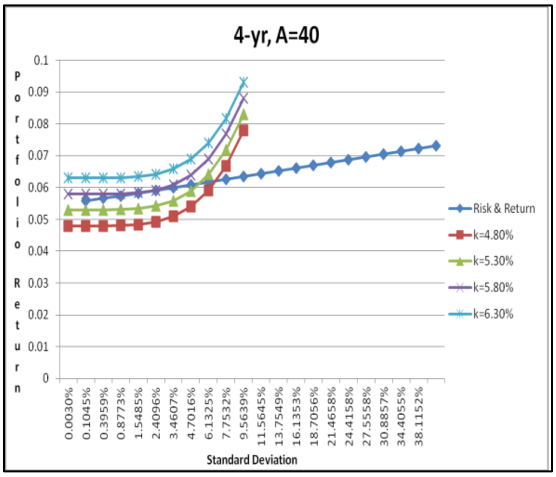


Fig: 9. Optimal Portfolio (n=5, A=4)

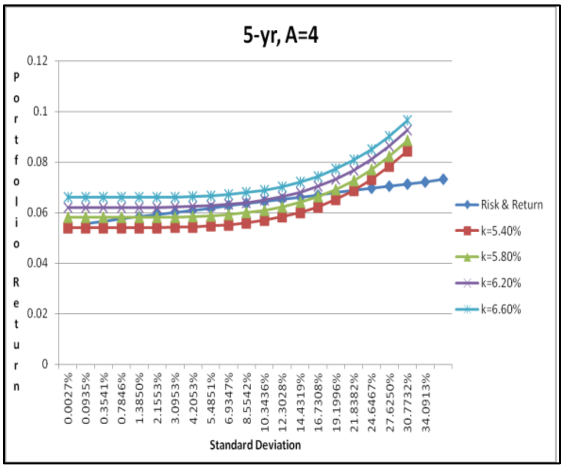
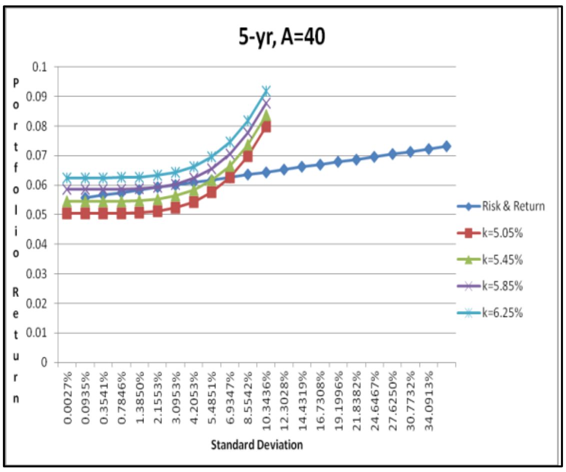


Fig: 10. Optimal Portfolio (n=5, A=40)



Thus, as the investment period of the investor increases from 1-year to 5-years, the risk taking capacity of the investor also increases. When the investment time horizon was 1-year, the optimal portfolio came out to be 80% bonds/20% stocks portfolio. On the other hand, when the investment time horizon was increased to 5-years, the optimal portfolio for the same investor became 55% bonds/ 45% stocks portfolio. Thus, with longer time horizon, the investor is willing to take a larger quantity of risk.

4.0 Based on Risk-Aversion Factor

Consider, for example, an investor with investment time horizon of 4-years. So, the utility function can be written as:

$$R_P = k + ½[(A\sigma^2_{R4}) / 4] \quad (15)$$

Case I: - Risk-Aversion Factor, A=4

$$R_P = k + (\sigma^2_{R4} / 2) \quad (16)$$

The curve with k=6.10% is tangent to the feasible set curve at approximately the portfolio which is 60% bonds and 40% stocks.

Case II: - Risk-Aversion Factor, A=40

$$R_P = k + (5*\sigma^2_{R4}) \quad (17)$$

In this case, the investor (with A=40) is less risk- tolerant (more risk-averse). This means that the investor is not willing to accept a larger amount of risk. So, the investor is trying to be conservative by selecting an optimal portfolio of less risky assets. Here, the feasible set is the same, and the efficient frontier is the same, but the indifference curves are noticeably less flatter. The optimal portfolio, which is 80% bonds and 20% stocks, lies at the point of tangency of the curve with k=5.80%.

5.0 Conclusion

With the investment time horizon increasing, the indifference curve starts becoming flatter. So, this implies that the investor is willing to accept a larger quantity of risk. With the investment period of the investor increasing from 1-year to 5-years, the risk taking capacity of the investor also increases. The choice of the optimal portfolio changed from 80% bond/20% stock portfolio to 55% bond/45% stock portfolio.

More risk-averse investors prefer the investment which has lower risk, while less risk-averse investors prefer the investment with a higher expected return. The more risk-averse an investor is, the lower will be the optimal portfolio on the return/risk spectrum defined by the efficient frontier. For very small values of the coefficient of risk aversion A (near zero), the investor is primarily concerned with maximizing expected return, and has little concern for risk. Conversely, for very large values of A, the investor is primarily concerned with minimizing risk. With A increasing from 4 to 40, the investor becomes less risk-tolerant (more risk-averse). This means that the investor is not willing to accept a larger amount of risk. So, the investor is trying to be conservative by selecting an optimal portfolio of less risky assets

Thus, no rational investor will invest in any portfolio unless its utility exceeds the Risk Free Rate. Investor will not opt for risky portfolios unless their returns exceed the risk free rate by an amount that is sufficient to overcome the risk scaled by a factor related to his risk-aversion factor.

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